Termination of Multifractal behaviour for Critical Disordered Dirac Fermions

J.-S. Caux[1], N. Taniguchi[1,2] and A. M. Tsvelik[1]
[1] *Department of Physics, University of Oxford, 1 Keble Road Oxford,OX1 3NP, UK*
[2] *Department of Physical Electronics, Hiroshima University, Higashi-Hiroshima 739, Japan*
(*Date* : February 1, 2008)

We consider Dirac fermions interacting with a disordered non-Abelian vector potential. The exact solution is obtained through a special type of conformal field theory including logarithmic correlators, without resorting to the replica or supersymmetry approaches. It is shown that the proper treatment of the conformal theory leads to a different multifractal scaling behaviour than initially expected. Moreover, the previous replica solution is found to be incorrect at the level of higher correlation functions.

Suggested PACS: 05.20-y, 02.50.Fz, 05.40.+j, 64.60.Ak

There exists a lot of evidence that the wavefunctions in disordered systems exhibit multifractal behaviour near the localization-delocalization transition (see for example [1–5]). Multifractality manifests itself in the anomalous scaling behaviour of moments of the local density of states (LDOS) with system size. This anomalous scaling is described, for a d-dimensional system in a box of size L with short-distance cutoff a, by the spectrum $\tau^*(q)$ of exponents defined as

$$\langle[\rho(\boldsymbol{r})]^q\rangle \sim (a/L)^{\tau^*(q)+d}; \tag{1}$$

Since $\tau^*(q)$ behaves linearly in q for a simple fractal (for example, a free wave in a box), the *nonlinear* dependence of $\tau^*(q)$ signals the multifractality of the critical wavefunctions. Such a behaviour has been observed in numerical calculations (for a review, see [3]) and a parabolic dependence, $\tau^*(q) = (q-1)(d-\alpha q)$, has been derived explicitly in a few different perturbative approaches such as the renormalization group (RG) one [1,5] and the optimal fluctuation-like methods [6].

There are strong restrictions on the allowable behaviour of $\tau^*(q)$ (see [3]) which require among other things the termination of the parabolic dependence for large enough q (otherwise one ends up with unphysical arbitrarily negative exponents in (1)). One can imagine that this termination should be achieved when all multiple-loop corrections are taken into account in a perturbative scheme. To check whether this hope is justified one inescapably needs to resort to nonperturbative methods.

In this respect two-dimensional disordered systems take a special place in investigating multifractality. On the one hand, there are *critical* disordered systems that are known to exhibit multifractality [3,4]: familiar examples being quantum Hall systems and disordered systems with spin-orbit interactions. On the other hand, in two dimensions one can study critical behaviour using the powerful nonperturbative machinery of conformal field theory (CFT) [7]. It is then natural to apply CFT to the problem of multifractality, and such attempts have indeed been initiated in Ref. [8,9] by exploring exactly solvable models of Dirac fermions (introduced in [10] in connection with the Integer Quantum Hall Effect) in Abelian or non-Abelian random vector fields. However, the results obtained in these papers are somewhat puzzling. In particular the solution to the problem with $SU(N)$ random vector field contains an infinite set of operators with *negative* conformal weights given by

$$h_q = \frac{q}{2} - \frac{N-1}{2N^2}(q^2+Nq) \tag{2}$$

leading via RG-type arguments to the set of (negative for large enough q) scaling exponents [9]

$$\tau^*(q) = (q-1)\left(2 - \frac{N-1}{N^2}q\right), \tag{3}$$

without explicit termination. The absence of the termination mechanism should be taken more seriously here than in the case of RG calculations, since this CFT result is nonperturbative and, if correct, exact.

We reveal in this letter the existence of a termination mechanism for the non-Abelian case, emerging straightforwardly from the correct full CFT treatment. This mechanism invalidates the result (2). For the sake of simplicity we illustrate this mechanism for the case $N = 2$, in which $\tau^*(q)$ becomes negative for the smallest value of q. Generalisation to arbitrary N is in principle possible.

The termination mechanism we present is different from the one in the *Abelian* random vector potential case, which have been successfully calculated in [11,12] by mapping into a random energy model or a type of Gaussian field theory. Moreover, for *normalized* wavefuntions, the termination of the multifractal spectrum can be expected to originate from the instability of the effective Liouville field theory [8] for large q.

The non-Abelian model has the additional virtue of being exactly solvable without using replicas or supersymmetry (SUSY) [13]. Therefore we also get a chance to independently check the results of these two approaches [9,14,15] comparing them with our exact results. It is

well known that the replica approach fails to reproduce some (off-critical) nonperturbative results like the level statistics in random matrix theory [16]. For critical disordered models one might expect more reliable results, since criticality implies universality. We here give an example demonstrating that this is not the case. Our exact solution shows that the higher correlation functions are identical with SUSY, but disagree with replicas.

Another interesting aspect of our exact solution is in the nature of the CFT involved. One of the landmarks of conventional CFTs is the power-law dependence of the physical correlators. In fact, in many instances, CFTs are identified by just such a property. Recently, however, it has been discovered that *logarithmic* dependence in physical correlators can appear in certain models outside of the usual class of so-called unitary minimal ones [17]. Logarithms are understood to be generated by degeneracies in the spectrum of conformal dimensions of the theory: when two operators have dimensions becoming degenerate, they metamorphose into a *logarithmic pair* with unconventional correlators involving not only powers, but logarithms. The model we are considering, in fact, enters this class of logarithmic CFTs. Moreover, as we shall see, it is the presence of such logarithmic operators that provides us with the solution to the problems associated to negative scaling dimensions.

Although the details of the calculation are somewhat involved, the conclusions can be relatively easily reached from our main result, which is the fusion rule (15). For the sake of clarity, though, we briefly outline in what follows the calculation process that leads to this result. Interested readers can find more extensive details in [18].

We consider N species of Dirac fermions living in a 2+1-dimensional space and interacting through a disordered vector potential A_μ transforming like the adjoint of an $su(N)$ algebra $\mathcal{A}$, to which they are coupled minimally. The disorder allows for hopping between the different species. Since the vector potential is time-independent, different Matsubara frequencies do not couple, and can be treated independently by a Euclidean two-dimensional theory with explicit frequency dependence. In fact, for a given realization of the disorder, the partition function takes the form of the fermionic path integral with the Dirac action

$$S[\Psi,\omega,A_\mu]=\int d^2x\bar{\Psi}(x)[\mathbf{I}\otimes\partial\!\!\!/-i\omega+i\,A\!\!\!/]\Psi \qquad (4)$$

(since we are in a two-dimensional Euclidean space, we take the Pauli matrices as Dirac γ matrices).

Arbitrary products of disorder-dependent single-particle Green's functions can then be calculated and averaged over the vector field distribution functional

$$P[A_\mu]=\frac{1}{\bar{g}}\int d^2x\,\mathrm{Tr}A_\mu(x)A_\mu(x) \qquad (5)$$

representing the usual δ-correlated Gaussian white noise for the random vector potential.

In the limits of infinite disorder strength $\bar{g}\to\infty$ and of vanishing frequency $\omega\to 0$, the theory becomes conformally invariant. Correlators can then be calculated according to the general principles behind CFT.

The derivation involves the following key moments. We separate the fermionic action into chiral parts, using holomorphic and antiholomorphic derivatives and fields ($2\partial=\partial_-, 2\bar{\partial}=\partial_+$), $A_\pm=A_1\pm iA_2$. It is important to note that now $A_\pm\in su^C(N)$, the complex extension $\mathcal{A}^C$ of $\mathcal{A}$. We then parametrize the vector fields by fields $g_\pm$ belonging to the complex extension G^C of the group $G=SU(N)$ as $A_\pm(x)=i\partial_\pm g_\pm(x)g_\pm^{-1}(x)$. The reality condition $A_+^\dagger(x)=A_-(x)$ translates into $g_+^\dagger(x)=g_-(x)$. We will use the notation $g_+(x)=g(x)$. This reparametrization induces a non trivial Jacobian in the path integral [9,19]

$$\mathcal{D}A_\mu=\mathcal{D}A_1\mathcal{D}A_2=\mathcal{D}G^C e^{2NW[g^\dagger g]} \qquad (6)$$

where $\mathcal{D}G^C$ is the Haar measure over $SU^C(N)$, and $W[g^\dagger g]$ is the Wess-Zumino-Novikov-Witten functional [20], here on level $k=-2N$, for the field combination $g^\dagger g$. This is a well-known functional for which correlators are in principle known [20].

The next step is to decouple the fermions from the random potential by the transformations

$$\Psi_\pm(x)\to g_\pm(x)\Psi'_\pm(x),\qquad \Psi_\pm^\dagger(x)\to {\Psi'_\pm}^\dagger(x)g_\pm^{-1}(x) \quad (7)$$

The Jacobian for this decoupling has the very important property of being proportional to the partition function at fixed disorder [19], thus cancelling it when computing the correlations for a given realization. This removes the need to invoke either the replica or SUSY methods to perform explicitly the disorder averaging.

Let us then consider correlators of the local operator

$$\mathcal{M}(z,\bar{z})=\mathrm{Tr}\bar{\Psi}\Psi={\Psi'_-}_a^\dagger h_{ab}\Psi'_{+b}+{\Psi'_+}_a^\dagger h_{ab}^{-1}\Psi'_{-b} \qquad (8)$$

(in which $h_{ab}=[g^\dagger g]_{ab}$, with th $SU(N)$ indices) which relates to the LDOS of the system and couples to the frequency perturbation in the original action. Notice that by doing this, we explicitly sum over the $SU(N)$ indices: the product $g^\dagger g$ is invariant under left-multiplication of g by $u\in SU(N)$. This is a crucial point: $g^\dagger g$ does not live on the complexified group manifold of $SU^C(N)$, but rather on the coset space $SU^C(N)/SU(N)$. The $SU(N)$ path-integration then simply factorizes out of the effective generating functional.

In general, when one deals with CFTs, physical states are associated to states in a highest-weight representation of the Virasoro algebra [7]. Quantum mechanical commutation rules are replaced in the radial quantization formalism by so-called fusion rules, which state the

short-distance singular behaviour of products of operators in the complex plane defined by the Euclidean spacetime variables $z = x_1 + ix_2, \bar{z} = x_1 - ix_2$. Such fusion rules are in principle completely obtainable from the four-point correlation functions of the physical operators, for which very powerful and extensive calculational methods are known.

One of the crucial aspects of the derivation is the proof that the WZNW model of level $k = -2N$ on the coset space $SU^C(N)/SU(N)$ actually carries a representation of the Virasoro algebra for $SU(N)$ for the same (analytically continued to negative) level. We refer the reader to [18] for an explicit proof of this statement.

The main procedures for obtaining the four-point coset correlators

$$H = \langle h_{a_1 b_1}(z_1, \bar{z}_1) h^{-1}_{b_2 a_2}(z_2, \bar{z}_2) h_{a_3 b_3}(z_3, \bar{z}_3) h^{-1}_{b_4 a_4}(z_4, \bar{z}_4) \rangle \quad (9)$$

and thus the fusion rules, in the case of WZNW models is very clearly set out in [20]. The "conformal bootstrap" provides differential equations (the Knizhnik-Zamolodchikov equations) for the so-called conformal blocks, whose solutions turn out to be hypergeometric functions of the variable z. The peculiarity that we encounter in the ensuing CFT, i.e. the one with $SU(2)$ Virasoro algebra on level $k = -4$, is that some conformal blocks *have logarithmic behaviour*. In fact, these conformal blocks read

$$\begin{aligned}
&\tilde{F}^a_1(z) = (1-z)F(3/2, 5/2; 2; z) \\
&\tilde{F}^b_1(z) = \frac{4}{\pi}\left[\frac{4/3}{z} + \tilde{F}^a_1(z)\ln z - 4/3 + (1-z)K_{11}(z)\right] \\
&\tilde{F}^a_2(z) = zF(3/2, 5/2; 3; z) \\
&\tilde{F}^b_2(z) = \frac{1}{2\pi}\left[\frac{16/3}{z} + \tilde{F}^a_2(z)\ln z - 4/3 + zK_{12}(z)\right]
\end{aligned} \quad (10)$$

where $F(a, b; c; z)$ is the hypergeometric function, and K_{11}, K_{12} are some functions regular as $z \to 0$.

Enforcing the locality conditions of quantum field theory then translates into "gluing" together the conformal blocks for the spacetime variables z and $\bar{z}$ in such a way that the combination is single-valued in the complex plane defined by the variable z (we refer the reader to [21] for the basic explanations). The main point that we want to stress here is that these gluing conditions in logarithmic CFTs are not, as was once thought, unsolvable, but rather a bit peculiar (for more details, see [18]). The crucial observation to make is that, as we take $z \to e^{2\pi i} z$, only the combination $\ln z\bar{z}$ remains invariant. This dictates the proper gluing procedure.

Leaving out the explicit details, we here simply provide the final expression for the coset correlator ($z = \frac{z_{12}z_{34}}{z_{13}z_{24}}, z_{ij} = z_i - z_j$):

$$H = |z_{13}z_{24}|^{3/2} \sum_{i,j=1,2} I_i \bar{I}_j H_{ij}(z, \bar{z}) \quad (11)$$

with

$$\begin{aligned}
H_{ij}(z, \bar{z}) &= \alpha |z(1-z)|^{3/2} [\tilde{F}^a_i(z)\tilde{F}^b_j(\bar{z}) + (a \leftrightarrow b)] \\
I_1 &= \delta_{a_1 a_2}\delta_{a_3 a_4} \qquad I_2 = \delta_{a_1 a_4}\delta_{a_2 a_3}
\end{aligned} \quad (12)$$

and for the resulting fusion rule between the coset operators (a similar one has been obtained also in Ref. [22]):

$$\begin{aligned}
&\mathrm{Tr}\, h(z_1, \bar{z}_1) h^{-1}(z_2, \bar{z}_2) \sim |z_{12}|^{3/2} \left[\frac{4}{3}[\frac{1}{z_{12}} A(z_2) + \right. \\
&\left. + \frac{1}{\bar{z}_{12}} \bar{A}(\bar{z}_2)] + [4\mathcal{I} + 2D(z_2, \bar{z}_2) + \ln|z_{12}| C(z_2, \bar{z}_2)]\right]
\end{aligned} \quad (13)$$

in which $\mathcal{I}$ is the identity operator. The correlators of the operators appearing in (13) are

$$\begin{aligned}
&\langle A(z_1)A(z_2)\rangle \sim z^2_{12}; \qquad \langle \bar{A}(\bar{z}_1)\bar{A}(\bar{z}_2)\rangle \sim \bar{z}^2_{12} \\
&\langle D(z_1, \bar{z}_1)D(z_2, \bar{z}_2)\rangle \sim -c_1 - \ln|z_{12}| \\
&\langle D(z_1, \bar{z}_1)C(z_2, \bar{z}_2)\rangle \sim 1; \\
&\langle C(z_1, \bar{z}_1)C(z_2, \bar{z}_2)\rangle = 0
\end{aligned} \quad (14)$$

where c_i are some constants, unimportant for our purposes.

There are many remarks that we can make from this unconventional Operator Product Expansion (OPE). First and foremost, wee notice that the most relevant operators appearing in the OPE (13), $A(z)$ and $\bar{A}(\bar{z})$, possess conformal weights $(-1, 0)$ and $(0, -1)$ respectively. Usually, the fusion rules for WZNW models [20] would imply that the adjoint operator, whose conformal weights are $(-1, -1)$, should appear in the OPE (13). But the term in the four-point function pointing to the presence of such an operator, does not appear since we are not allowed, by the requirement of single valuedness, to multiply the logarithmic solutions in the holomorphic and antiholomorphic sectors together. This requirement cannot be seen from the chiral conformal algebra studied in [9]: it can only come out of the solution for the full correlator that we have obtained.

The four-point $\mathcal{M}$ correlator can be calculated along the same lines as the coset correlator (11) by including the free fermion contributions and contracting the indices appropriately. This four-point function allows us to extract the OPE of $\mathcal{M}$ with itself, which in turn determines the scaling of the local moments. We find that the correct OPE reads [18]:

$$\mathcal{M}(1)\mathcal{M}(2) \sim \frac{1}{|z_{12}|^{1/2}}\left[\mathcal{I} + D(2) + \frac{1}{2}\ln|z_{12}| C(2) + ...\right] \quad (15)$$

where α is some constant. The crucial fact is that the operator in the symmetric representation does not appear

here again, like in (13). D and C operators fuse as (14), whereas $\mathcal{M}, D$ and C fuse as

$$D(1)\mathcal{M}(2) \sim [\ln|z_{12}| + c_2]\mathcal{M}(2) + ...$$
$$C(1)\mathcal{M}(2) \sim \mathcal{M}(2) + ... \qquad (16)$$

The fusion rules (15,16) constitute the most important result of our paper. The remarkable feature of these OPEs is the appearance of logarithmic operators D and C. Their origin lies in logarithmic singularities present in multi-point correlation functions. These operators do not form the usual diagonalizable representations of the Virasoro energy operator L_0 [17] which is evident from their unconventional correlation functions (14).

The first comment that we can make about the OPE (15) is that it invalidates the previous replica method treatment [14,15], for which the corresponding OPE read

$$\mathcal{M}(1)\mathcal{M}(2) \sim \frac{1}{|z_{12}|^{1/2}}\,[\mathcal{I} + z[D(2) + \ln|z|C(2)] + \\ + \bar{z}[\bar{D}(2) + \ln|z|\bar{C}(2)] + ...] \qquad (17)$$

in which $C, D, \bar{C}, \bar{D}$ were chiral logarithmic pairs with conformal dimensions (1,0) and (0,1) respectively. This is different from the pairs introduced in (14) which have dimensions (0,0). Even though the two-point functions coincide in both treatments, the higher-point functions are different. Operator dimensions are calculated correctly by replicas, but correlators of higher order are not.

Our result (15) will also lead to a markedly different behaviour than the one associated to equation (3). Since the LDOS ρ is related to the imaginary part of $\mathcal{M}$, its local moments ρ^q will scale at most like the *most relevant part* of $\mathcal{M}^q$, which can be obtained by point-splitting from (15). We can draw the following comparison table between our results for the conformal weights h_q of the most relevant operator contained in $\mathcal{M}^q$ as a function of the power q, and the ones obtained in the previous treatment [9]:

q	1	2	3	4	...
h_q ([9])	1/8	-1	$-9/4$	-4	...
h_q (exact)	1/8	0	1/8	0	...

(18)

The parabolic series of negative exponents is seen to be cut right from the beginning for $SU(2)$. Note that logarithms are weight-zero objects, so in fact they appear implicitly in our weights above for all q. The presence of these logarithmic prefactors makes the usual treatment with τ^* exponents somewhat unsatisfactory, since it can only describe pure power scaling and not the logarithmic corrections to it that we have found.

In conclusion, we have shown that moments of the local density operators in the problem of non-Abelian randomness obey different scaling relations than expected [9], that do not fit in the standard multifractal description. The negative-dimensional operators are suppressed by the presence of logarithmic operators in the relevant OPEs. Moreover, although replicas give correct primary field scaling dimensions, they fail to reproduce the detailed form of the correlators. Further details can be found in [18].

We are grateful to I. Kogan, I. Lerner, K. Efetov, A. Gogolin and V. Kravtsov for interesting discussions and interest to the work. J.-S. C. acknowledges support from NSERC Canada, and from the Rhodes Trust. N. T. acknowledges support from Japan Society for the Promotion of Science.

[1] F. Wegner, Z. Phys. B **36**, 209 (1980).
[2] H. Aoki, J. Phys. C16, L205 (1983).
[3] W. Pook and M. Janssen, Z. Phys. B**82**, 295 (1991); M. Janssen, Int. J. Mod. Phys. B **8**, 943 (1994).
[4] B. Huckestein, Rev. Mod. Phys. 67, 357 (1995).
[5] B. L. Altshuler, V. E. Kravtsov and I. V. Lerner, in *Mesoscopic Phenomena in Solids* (1991) Elsevier Science p.449.
[6] B. A. Muzykantskii and D. E. Khmelnitskii, Phys. Rev. B**51**, 5480 (1995); V. Fal'ko and K. B. Efetov, Phys. Rev. B**52**, 17413 (1995); I. E. Smolyarenko and B. L. Altshuler, cond-mat/9606181.
[7] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. **B241**, 333 (1984).
[8] I. I. Kogan, C. Mudry, and A. M. Tsvelik, Phys. Rev. Lett. **77**, 707 (1996).
[9] C. Mudry, C. Chamon, and X.-G. Wen, Nucl. Phys. **B466[FS]**, 383 (1996).
[10] A. W. W. Ludwig *et al*, Phys. Rev. B **50**, 7526 (1994).
[11] C. Chamon, C. Mudry and X.-G. Wen, Phys. Rev. Lett. **77**, 4194 (1996).
[12] H. E. Castillo *et al*, cond-mat/9706084.
[13] K. B. Efetov, Adv. Phys. **32**, 53 (1983); see also *Supersymmetry in Disorder and Chaos* (Cambridge University Press, New York, 1997).
[14] A. A. Nersesyan, A. M. Tsvelik, and F. Wenger, Phys. Rev. Lett. **72**, 2628 (1994), Nucl. Phys. **B438[FS]**, 561 (1995).
[15] J.-S. Caux, I. I. Kogan, and A. M. Tsvelik, Nucl. Phys. **B466[FS]**, 444 (1996).
[16] J. J. M. Verbaarschot and M. R. Zirnbauer, J. Phys. A **17**, 1093 (1985).
[17] V. Gurarie, Nucl. Phys. **B410** (1993) 535.
[18] J.-S. Caux, N. Taniguchi and A. M. Tsvelik, (to be published).
[19] D. Bernard, hep-th/9509137 (unpublished).
[20] V. G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. **B247**, 83 (1984).
[21] Vl. S. Dotsenko and V. A. Fateev, Nucl. Phys. **B240**, 312 (1984).
[22] J.-S. Caux, A. Lewis, I. I. Kogan and A. M. Tsvelik, Nucl. Phys. B**489**, 469 (1997).